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Conic positive definiteness and sharp minima of fractional orders in vector optimization problems ☆

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ABSTRACT

Motivated by the fact that the usual positive definiteness does not work in an infinite space, we introduce the concept of S-positive definiteness with respect to an ordering cone in a general Banach space and show that the S-positive definiteness plays the same role as the usual positive definiteness in the finite dimensional case. As applications, we study sharp and weak sharp minima of fractional orders in vector optimization.

1. Introduction

Let X be a Banach space and $\phi : X \to \mathbb{R}$ be a $2n$-time smooth function, where n is a natural number. The following result is well known and useful in optimization (cf. [14, Proposition 5.2]).

Proposition I. *If X is finite dimensional, $\phi^{(k)}(\bar{x}) = 0$ for $k = 1, \dots, 2n-1$ and $\phi^{(2n)}(\bar{x})$ is positively definite (i.e., $\phi^{(2n)}(\bar{x})(h^{2n}) > 0$ for all $h \in X \setminus \{0\}$ where $h^{2n} = (h, \dots, h)$), then $\bar{x}$ is a local minimizer of ϕ.*

The following example shows that Proposition I is not necessarily true when X is infinite dimensional.

Example 1.1. Let $X = l^2$ and

$$\phi(x) := \sum_{k=1}^{\infty}\left(\frac{x_k^2}{k^3} - x_k^4\right), \quad \forall x = (x_1, x_2, \dots) \in l^2.$$

Let $\bar{x} = 0$ be the zero element in l^2. It is easy to verify that $\phi'(\bar{x}) = 0$ and $\phi''(\bar{x})(x^2) = \sum_{k=1}^{\infty} \frac{2x_k^2}{k^3} > 0$ for all $x \in l^2 \setminus \{0\}$. But $\bar{x}$ is not a local minimizer of ϕ. Indeed, if this is not the case, there exists $\delta > 0$ such that

$$0 = \phi(\bar{x}) \leqslant \min_{x \in B(\bar{x}, \delta)} \phi(x), \tag{1.1}$$

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where $B(\bar{x},\delta) := \{x \in X\colon \ \|x-\bar{x}\| < \delta\}$. For each natural number k, let $x(k)$ be an element in l^2 such that the k-th component is $\frac{1}{k}$ and all other components are 0. Then $x(k) \to 0 = \bar{x}$ and $\phi(x(k)) = \frac{1}{k^5} - \frac{1}{k^4} < 0$ (for all $k \geqslant 2$). This contradicts (1.1). Hence $\bar{x}$ is not a local minimizer of ϕ.

It is more complicated to consider the vector-valued function setting. In this paper, we mainly study vector optimization problems in infinite dimensional Banach spaces, which has been found to have important applications in many fields such as economics, management science and engineering (see [1,5,7–9,13,15,20]). One of our aims is to establish sufficiency results similar to Proposition I for a vector optimization problem when the objective is a $2n$-time smooth function between two infinite dimensional Banach spaces. To do this, in view of Example 1.1, we introduce the concept of strong positive definiteness with respect to vector preorder induced by a closed and convex cone. Our strong positive definiteness reduces to the positive definiteness in the finite dimensional setting. In terms of the strong positive definiteness, Proposition I is extended to vector optimization problems in infinite dimensional spaces.

For a real-valued function $\phi: X \to \mathbb{R}$ on a Banach space, recall that $a \in X$ is a sharp minimum of ϕ if there exist $\eta, \delta \in (0,+\infty)$ such that

$$\eta\|x-a\| \leqslant \phi(x) - \phi(a) \quad \forall x \in B(a,\delta). \tag{1.2}$$

The notion of a sharp minimum, equivalently, a strong isolated minimizer or strongly unique local minimum, was introduced by Polyak (see [17] and references therein). It has far-reaching consequences for convergence analysis in mathematical programming (see [3,12,16] and references therein). As a generalization of sharp minima, Ferris [6] introduced and studied weak sharp minima for real-valued functions. Extending the sharp minima in the sense of (1.2) to the multiobjective optimization, Jimenez [10,11] introduced and studied the strict Pareto efficiency in multiobjective programming. Using the technique of variational analysis, in terms of the normal cone and coderivative, Zheng et al. [21] established some characterizations of the sharp minima for nonsmooth vector optimization problems. Deng and Yang [4] considered weak sharp minima for linear vector optimization problems in Euclidean spaces. In this paper, we consider sharp minima for smooth vector optimization problems in Banach spaces. In particular, using the strong positive definiteness, we provide some results on sharp minimum property of Pareto solutions and ideal solutions for smooth vector optimization problems. We also consider quadratic vector optimization problems in Banach spaces. We establish sharper results on sharp minima and weak sharp minima for such problems.

2. Preliminaries

Throughout this paper, all spaces considered are real Banach spaces. Let Y be a Banach space and Y^* be its dual space. Let $C \subset Y$ be a closed and convex cone with $\mathrm{int}(C) \neq \emptyset$ and C^+ denote the dual cone of C, that is, $C^+ = \{y^* \in Y^*\colon \ 0 \leqslant \langle y^*, c\rangle \ \forall c \in C\}$. For $y_1, y_2 \in Y$, define $y_1 <_C y_2$ and $y_1 \leqslant_C y_2$ as $y_2 - y_1 \in \mathrm{int}(C)$ and $y_2 - y_1 \in C$, respectively.

Let A be a subset of Y and $a \in A$. We say that

(i) a is a weak Pareto efficient point of A if there exists no point $y \in A \setminus \{a\}$ such that $y <_C a$;
(ii) a is a Pareto efficient point of A if there exists no point $y \in A \setminus \{a\}$ such that $y \leqslant_C a$;
(iii) a is an ideal point of A if $a \leqslant_C y$ for all $y \in A$.

Let $\mathrm{WE}(A,C)$ and $\mathrm{E}(A,C)$ denote the set of all weak Pareto efficient points of A and the set of all Pareto efficient points of A, respectively. It is clear that

$$a \in \mathrm{WE}(A,C) \quad \Leftrightarrow \quad \big(a - \mathrm{int}(C)\big) \cap A = \emptyset$$

and

$$a \in \mathrm{E}(A,C) \quad \Leftrightarrow \quad (a - C) \cap A = \{a\}.$$

Let f be a function between Banach spaces X and Y and consider the following vector optimization problem

$$C - \min_{x\in X} f(x). \tag{2.1}$$

A vector $\bar{x} \in X$ is called a local weak Pareto solution (resp. Pareto solution) of (2.1) if there exists $\delta > 0$ such that $f(\bar{x})$ is a weak Pareto efficient point (resp. Pareto efficient point) of $f(B(\bar{x},\delta))$. We say that $\bar{x}$ is a local ideal solution of (2.1) if there exists $\delta > 0$ such that $f(\bar{x})$ is an ideal point of $f(B(\bar{x},\delta))$. We say that $\bar{x}$ is a global ideal solution of (2.1) if $f(\bar{x})$ is an ideal point of $f(X)$.

We say that $\bar{x}$ is a sharp Pareto solution of (2.1) of order r if there exist $\eta, \delta \in (0,+\infty)$ such that

$$\eta\|x-\bar{x}\| \leqslant d\big(f(x) - f(\bar{x}), -C\big)^r \quad \forall x \in B(\bar{x},\delta),$$

where $d(y,-C) := \inf\{\|y-z\|\colon \ z \in -C\}$.

Recall that a mapping $f : X \to Y$ is C-convex if

$$f\big(tx_1 + (1-t)x_2\big) \leqslant_C tf(x_1) + (1-t)f(x_2) \quad \forall x_1, x_2 \in X \text{ and } \forall t \in [0,1].$$

The following lemma is known and useful in optimization (cf. [22, Theorem 3.2], the remarks after the proof of [22, Theorem 3.2] and [9, Theorem 5.13]).

Lemma 2.1. *Let $f : X \to Y$ be a smooth mapping and $\bar{x} \in X$. Then, the following statements hold*:

(i) *If there exists $c^* \in C^+$ with $\|c^*\| = 1$ such that*

$$c^* \circ f'(\bar{x}) = 0, \tag{2.2}$$

then $\bar{x}$ is a weak Pareto solution of (2.1)*.*

(ii) *If f is C-convex and $\bar{x}$ is a weak Pareto solution of* (2.1)*, then there exists $c^* \in C^+$ with $\|c^*\| = 1$ such that* (2.2) *holds.*

Proposition 2.2. *Let $f : X \to Y$ be a smooth mapping and $\bar{x}$ be a point in X. Consider the following statements*:

(i) *$\bar{x}$ is a local ideal solution of* (2.1)*.*
(ii) *$f'(\bar{x})(h) \in C \cap -C$ for all $h \in X$.*
(iii) *$\bar{x}$ is a global ideal solution of* (2.1)*.*

Then (i) $\Rightarrow$ (ii)*. If, in addition, f is C-convex, then* (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii)*.*

Proof. Suppose that (i) holds, and let h be an arbitrary point in X. Then there exists $\delta > 0$ such that $f(\bar{x}) \leqslant_C f(\bar{x} + th)$ for all $t \in (0, \delta)$. Hence, $\frac{f(\bar{x}+th)-f(\bar{x})}{t} \in C$ for all $t \in (0, \delta)$. Since C is closed, it follows that $f'(\bar{x})(h) = \lim_{t\to 0} \frac{f(\bar{x}+th)-f(\bar{x})}{t} \in C$. By the arbitrariness of h, one has $-f'(\bar{x})(h) = f'(\bar{x})(-h) \in C$. This shows that (ii) holds.

Next we assume that f is C-convex. To prove (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii), it suffices to show (ii) $\Rightarrow$ (iii). To do this, suppose that (ii) holds. Let $x \in X$. By the C-convexity of f, one has

$$f\big(\bar{x} + t(x - \bar{x})\big) \leqslant_C (1-t)f(\bar{x}) + tf(x) \quad \forall t \in (0,1],$$

that is,

$$\frac{f(\bar{x} + t(x - \bar{x})) - f(\bar{x})}{t} \leqslant_C f(x) - f(\bar{x}) \quad \forall t \in (0,1].$$

Hence, $f'(\bar{x})(x - \bar{x}) \leqslant_C f(x) - f(\bar{x})$. It follows from (ii) that $f(\bar{x}) \leqslant_C f(x)$. This shows that (iii) holds. The proof is completed. $\square$

3. The conic positive definiteness of multilinear mappings

In the remainder of this paper, let X, Y be Banach spaces and let Y be equipped with a preorder induced by a closed and convex cone C in Y with $\mathrm{int}(C) \neq \emptyset$ and $C \neq Y$. For a natural number n, let the product space $X^n := \{(x_1, \dots, x_n)\colon x_i \in X, i = 1, \dots, n\}$ be equipped with the norm $\|(x_1, \dots, x_n)\| = \sum_{i=1}^n \|x_i\|$.

Let $\Phi : X^n \to Y$ be an n-linear and symmetric mapping, that is, for any $s, t \in \mathbb{R}$ and $x_1, z_1, x_2, \dots, x_n \in X$

$$\Phi(sx_1 + tz_1, x_2, \dots, x_n) = s\Phi(x_1, x_2, \dots, x_n) + t\Phi(z_1, x_2, \dots, x_n)$$

and

$$\Phi(x_1, \dots, x_n) = \Phi(x_{i_1}, \dots, x_{i_n}),$$

where $(i_1, \dots, i_n)$ is an arbitrary permutation of $(1, \dots, n)$. For each $x \in X$, let $\Phi(x^n) := \Phi(x, \dots, x)$. Let $f : X \to Y$ be a mapping. It is known that $f^{(n)}(x)$ is an n-linear, symmetric and continuous mapping if f is n-time smooth.

Definition 3.1. Let $\Phi : X^n \to Y$ be an n-linear symmetric mapping.

(i) Φ is said to be positively definite (resp. positively semi-definite) with respect to the ordering cone C if

$$0 <_C \Phi\big(x^n\big) \quad \big(\text{resp. } 0 \leqslant_C \Phi\big(x^n\big)\big) \quad \text{for all } x \in X \setminus \{0\};$$

(ii) Φ is said to be S-positively definite with respect to the ordering cone C if there exists $\eta > 0$ such that

$$\Phi\left(x^n\right) + \eta B_Y \subset C \quad \text{for all } x \in X \text{ with } \|x\| = 1,$$

where B_Y denotes the unit ball of Y.

If n is an odd number and $\Phi : X^n \to Y$ is an n-linear symmetric mapping, $\Phi((-h)^n) = -\Phi(h^n)$ for each $h \in X$; thus, under the assumption that the ordering cone is pointed (i.e., $C \cap -C = \{0\}$), Φ is positively semi-definite if and only if $\Phi = 0$. By the separation theorem, it is easy to verify that an n-linear symmetric mapping $\Phi : X^n \to Y$ is positively semi-definite with respect to the ordering cone C if and only if the composite $c^* \circ \Phi$ is a positively semi-definite for any $c^* \in C^+$.

Remark 3.1. For a mapping $f : X \to Y$, noting that f is C-convex if and only if $c^* \circ f$ is convex for all $c^* \in C^+$, one can see that a twice differentiable function f is C-convex if and only if the second derivative $f''(x)$ is positively semi-definite for each $x \in X$.

Clearly, an n-linear symmetric mapping Φ is positively definite whenever it is S-positively definite. The following proposition shows that the converse is also true when X is finite dimensional.

Proposition 3.2. *Let X be finite dimensional and $\Phi : X^n \to Y$ be an n-linear and symmetric mapping. Then Φ is S-positively definite with respect to C if and only if Φ is positively definite with respect to C.*

Proof. The necessity part is trivial. To prove the sufficiency part, suppose to the contrary that for every natural number k there exist $x_k \in X$ and $y_k \in B_Y$ such that $\|x_k\| = 1$ and

$$\Phi\left(x_k^n\right) + \frac{1}{k} y_k \notin C. \tag{3.1}$$

Since X is finite dimensional, without loss of generality we can assume that $x_k \to x_0$ (passing to a subsequence if necessary). Then $\|x_0\| = 1$, and so $\Phi(x_0^n) \in \operatorname{int}(C)$. Take a constant $r > 0$ such that $\Phi(x_0^n) + rB_Y \subset C$. By the continuity of Φ, one has that $\Phi(x_k^n) + \frac{1}{k} y_k \in \Phi(x_0^n) + rB_Y$ for all k sufficiently large, contradicting (3.1). The proof is completed. □

In the case when X is infinite dimensional, Proposition 3.2 is not necessarily true. For example, let $n = 2$, $X = l^2$, $Y = R$, $C = R_+$ and $\Phi(u, v) = \sum_{k=1}^{\infty} \frac{u_k v_k}{k}$ for any $u = (u_1, u_2, \ldots), v = (v_1, v_2, \ldots) \in l^2$. Clearly, Φ is bilinear and symmetric, and $0 < \Phi(x, x)$ for any $x \in l^2 \setminus \{0\}$. However, Φ is not S-positively definite with respect to R_+. In fact, for every natural number k, let $e(k)$ denote the element such that the k-th component of $e(k)$ is 1 and all other components are 0. Then $\|e(k)\| = 1$ and $\Phi(e(k)^2) = \frac{1}{k}$. Let η be an arbitrary positive number. Then, $\Phi(e(k)^2) - \frac{\eta}{2} \notin R_+$ for all k large enough, but $\Phi(e(k)^2) - \frac{\eta}{2} \in \Phi(e(k)^2) + \eta B_R$ for all k. This shows that Φ is not S-positive definite with respect to R_+.

Now we provide an example of an S-positively definite mapping between two infinite dimensional spaces. Let $X = l^2$ and Y be an infinite dimensional Banach space ordered by a closed and convex cone with $\operatorname{int}(C) \neq \emptyset$. Take an element $c_0 \in \operatorname{int}(C)$ and $r > 0$ such that $c_0 + 2rB_Y \subset C$. Let $\{y_n\}$ be a sequence in rB_Y and define $\Phi : l^2 \times l^2 \to Y$ to be such that

$$\Phi(u, v) = \sum_{n=1}^{\infty} u_n v_n (c_0 + y_n) \quad \text{for all } u = (u_1, u_2, \ldots), v = (v_1, v_2, \ldots) \in l^2.$$

It is clear that Φ is a bilinear symmetric mapping. For any $x = (x_1, x_2, \ldots) \in l^2$ with $\|x\| = 1$, $\Phi(x^2) = c_0 + \sum_{n=1}^{\infty} x_n^2 y_n$ and so

$$\left\|\Phi\left(x^2\right) - c_0\right\| \leqslant \sum_{n=1}^{\infty} x_n^2 \|y_n\| \leqslant \sum_{n=1}^{\infty} x_n^2 r = r;$$

hence

$$\Phi\left(x^2\right) + rB_Y = c_0 + \Phi\left(x^2\right) - c_0 + rB_Y \subset c_0 + 2rB_Y \subset C.$$

This shows that Φ is S-positively definite.

In terms of the concept of S-positive definiteness, we provide high order sufficient conditions for $\bar{x}$ to be a local or global sharp Pareto solution of vector optimization problem (2.1).

Theorem 3.3. *Let f be a mapping between Banach spaces X and Y. Let $\bar{x} \in X$ and n be a natural number such that f is $2n$-time differentiable around $\bar{x}$ and $f^{(2n)}(\bar{x})$ is S-positively definite. Suppose that there exists $c^* \in C^+$ with $\|c^*\| = 1$ such that*

$$c^* \circ f^{(i)}(\bar{x}) = 0, \quad i = 1, \ldots, 2n-1. \tag{3.2}$$

Then $\bar{x}$ is a local Pareto solution of (2.1) and there exist $\eta, \delta \in (0, +\infty)$ such that

$$\eta\|x-\bar{x}\| \leqslant d\big(f(x)-f(\bar{x}), -C\big)^{\frac{1}{2n}} \quad \text{for all } x \in B(\bar{x}, \delta). \tag{3.3}$$

If, in addition, f is C-convex, then $\bar{x}$ is a global Pareto solution of (2.1) and there exists $\eta_0 \in (0, +\infty)$ such that

$$\eta_0\|x-\bar{x}\| \leqslant \max\big\{d\big(f(x)-f(\bar{x}), -C\big)^{\frac{1}{2n}}, d\big(f(x)-f(\bar{x}), -C\big)\big\} \quad \text{for all } x \in X. \tag{3.4}$$

Proof. Since $f^{(2n)}(\bar{x})$ is S-positively definite with respect to C, there exists $\eta' > 0$ such that

$$\frac{1}{(2n)!} f^{(2n)}(\bar{x})\big(h^{2n}\big) + 2\eta' B_Y \subset C \quad \text{for all } h \in X \text{ with } \|h\| = 1.$$

Noting that $c^* \in C^+$ and $\|c^*\| = 1$, it follows that

$$\frac{1}{(2n)!}\big(c^* \circ f^{(2n)}(\bar{x})\big)\big(h^{2n}\big) \geqslant 2\eta'\|h\|^{2n} \quad \text{for all } h \in X. \tag{3.5}$$

Let $\phi(x) := \langle c^*, f(x)\rangle$ for all $x \in X$. Then ϕ is $2n$-time differentiable at $\bar{x}$. Hence,

$$\phi(x) = \phi(\bar{x}) + \sum_{i=1}^{2n} \frac{1}{i!}\phi^{(i)}(\bar{x})\big((x-\bar{x})^i\big) + o\big(\|x-\bar{x}\|^{2n}\big).$$

It follows that there exists $\delta > 0$ such that

$$\phi(x) - \phi(\bar{x}) - \sum_{i=1}^{2n} \frac{1}{i!}\phi^{(i)}(\bar{x})\big((x-\bar{x})^i\big) \geqslant -\eta'\|x-\bar{x}\|^{2n} \quad \forall x \in B(\bar{x}, \delta).$$

Since $\phi^{(i)}(\bar{x}) = c^* \circ f^{(i)}(\bar{x})$, it follows from (3.2) and (3.5) that

$$\eta'\|x-\bar{x}\|^{2n} \leqslant \phi(x) - \phi(\bar{x}) \quad \forall x \in B(\bar{x}, \delta).$$

Since

$$\phi(x) - \phi(\bar{x}) = \big\langle c^*, f(x) - f(\bar{x})\big\rangle \leqslant \big\langle c^*, f(x) - f(\bar{x}) + c\big\rangle \leqslant \big\|f(x) - f(\bar{x}) + c\big\|$$

for any $c \in C$, $\phi(x) - \phi(\bar{x}) \leqslant d(f(x) - f(\bar{x}), -C)$. Thus, one sees that (3.3) holds with $\eta = \eta'^{\frac{1}{2n}}$. Let $x \in B(\bar{x}, \delta)$ be such that $f(x) \leqslant_C f(\bar{x})$. Then $d(f(x) - f(\bar{x}), -C) = 0$. It follows from (3.3) that $x = \bar{x}$ and hence $f(x) = f(\bar{x})$. This shows that $\bar{x}$ is a local Pareto solution of (2.1).

Next suppose that f is C-convex. Then, it is easy to verify that the function $x \mapsto d(f(x) - f(\bar{x}), -C)$ is convex. This and (3.3) imply that for any $x \in X \setminus B(\bar{x}, \delta)$,

$$\begin{aligned}
\eta\delta' &\leqslant d\left(f\left(\bar{x} + \delta'\frac{x-\bar{x}}{\|x-\bar{x}\|}\right) - f(\bar{x}), -C\right)^{\frac{1}{2n}} \\
&\leqslant \left(\left(1 - \frac{\delta'}{\|x-\bar{x}\|}\right)d\big(f(\bar{x}) - f(\bar{x}), -C\big) + \frac{\delta'}{\|x-\bar{x}\|}d\big(f(x) - f(\bar{x}), -C\big)\right)^{\frac{1}{2n}} \\
&= \left(\frac{\delta'}{\|x-\bar{x}\|}d\big(f(x) - f(\bar{x}), -C\big)\right)^{\frac{1}{2n}},
\end{aligned}$$

where δ' is an arbitrary constant in $(0, \delta)$. Hence,

$$\eta^{2n}\delta'^{2n-1}\|x-\bar{x}\| \leqslant d\big(f(x) - f(\bar{x}), -C\big) \quad \forall x \in X \setminus B(\bar{x}, \delta).$$

Letting $\eta_0 := \min\{\eta, \eta^{2n}\delta'^{2n-1}\}$, it follows from (3.3) that (3.4) holds. The proof is completed. □

It is natural to ask whether $\max\{[f(x) - f(\bar{x})]_+^{\frac{1}{2n}}, [f(x) - f(\bar{x})]_+\}$ in (3.4) of Theorem 3.3 can be replaced by either $[f(x) - f(\bar{x})]_+^{\frac{1}{2n}}$ or $[f(x) - f(\bar{x})]_+$, that is, whether there exists $\eta > 0$ under the C-convexity assumption on f such that one of the following (3.6) and (3.7) holds:

$$\eta\|x-\bar{x}\| \leqslant d\big(f(x) - f(\bar{x}), -C\big)^{\frac{1}{2n}} \quad \text{for all } x \in X, \tag{3.6}$$

and

$$\eta\|x-\bar{x}\| \leqslant d\big(f(x) - f(\bar{x}), -C\big) \quad \text{for all } x \in X. \tag{3.7}$$

Example 3.1. Let $n=1$, $X=Y=\mathbb{R}$, $C=\mathbb{R}_+$, and $f(x)=x^2$ if $|x|\leqslant 1$ and $f(x)=2|x|-1$ otherwise. Then f is a continuous convex function and it is also twice Frechet differentiable around 0; moreover $f'(0)=0$ and $f''(0)=2$. Since $\lim_{x\to\infty}\frac{(f(x)-f(0))^{\frac{1}{2}}}{|x-0|}=0$, (3.6) does not hold for each $\eta>0$.

Example 3.2. Let $n=1$, $X=Y=\mathbb{R}$, $C=\mathbb{R}_+$, and $f(x)=x^2$ for all $x\in\mathbb{R}$. Then f is a twice Frechet differentiable convex function on $\mathbb{R}$ such that $f'(0)=0$ and $f''(0)=2$. Since $\lim_{x\to 0}\frac{f(x)-f(0)}{|x-0|}=0$, (3.7) does not hold for each $\eta>0$.

We say that $\bar{x}\in X$ is a local sharp ideal solution of order $\gamma\in(0,+\infty)$ for (2.1) if there exist $\eta,\delta\in(0,+\infty)$ such that

$$\eta\|x-\bar{x}\|\leqslant d\big(f(x)-f(\bar{x}),-C\big)^{\gamma}\quad\forall x\in B(\bar{x},\delta),$$

and

$$f(\bar{x})\leqslant_C f(x)\quad\forall x\in B(\bar{x},\delta).\tag{3.8}$$

With (3.2) replaced by a stronger assumption, we have the following sufficient condition for sharp ideal solutions of (2.1).

Theorem 3.4. *Let f be a mapping between Banach spaces X and Y. Let $\bar{x}\in X$ and n be a natural number such that f is $2n$-time differentiable around $\bar{x}$ and $f^{(2n)}(\bar{x})$ is S-positively definite. Suppose that*

$$f^{(i)}(\bar{x})=0,\quad i=1,\dots,2n-1.\tag{3.9}$$

Then there exist $\eta,\delta\in(0,+\infty)$ such that (3.3) *and* (3.8) *hold. Consequently, $\bar{x}$ is a local sharp ideal solution of order $\frac{1}{2n}$ for* (2.1)*.*

Proof. By Theorem 3.3, we need only show that there exists $\delta>0$ such that (3.8) holds. From the S-positive definiteness of $f^{(2n)}(\bar{x})$, it is easy to verify that there exists $\eta>0$ such that

$$\frac{1}{(2n)!}f^{(2n)}(\bar{x})\big(x^{2n}\big)+\eta\|x\|^{2n}B_Y\subset C\quad\forall x\in X.\tag{3.10}$$

On the other hand, by (3.9), one has

$$f(x)=f(\bar{x})+\frac{1}{(2n)!}f^{(2n)}(\bar{x})\big((x-\bar{x})^{2n}\big)+o\big(\|x-\bar{x}\|^{2n}\big).$$

Hence there exists $\delta>0$ such that

$$f(x)-f(\bar{x})\in\frac{1}{(2n)!}f^{(2n)}(\bar{x})\big((x-\bar{x})^{2n}\big)+\eta\|x-\bar{x}\|^{2n}B_Y\quad\forall x\in B(\bar{x},\delta).$$

This and (3.10) imply that (3.8) holds. The proof is completed. □

4. Quadratic vector optimization

In this section, we consider quadratic vector optimization problems in general Banach spaces. We say that a vector-valued function f between Banach spaces X and Y is quadratic if there exist a bilinear symmetric and continuous mapping $\Phi: X\times X\to Y$, a continuous linear operator $T: X\to Y$ and a point b in Y such that

$$f(x)=\Phi\big(x^2\big)+T(x)+b\quad\forall x\in X.\tag{4.1}$$

In the case when $Y=\mathbb{R}$, it is well known (cf. [18, Theorem 4.5]) that a quadratic function f defined by (4.1) is convex if and only if Φ is positively semi-definite. In the case when Y is a general Banach space, it is easy from Remark 3.1 to verify that the corresponding result holds; more precisely, a quadratic function f defined by (4.1) is convex with respect to the ordering cone C in Y if and only if Φ is positively semi-definite with respect to C.

Let $a_j^*\in X^*$, $c_j\in\mathbb{R}$ and $P:=\{x\in X\colon\ \langle a_j^*,x\rangle-c_j\leqslant 0,\ \ j=1,\dots,k\}$. Consider the following quadratic convex vector optimization problem:

$$C-\min_{x\in P}\big(\Phi\big(x^2\big)+T(x)+b\big).\tag{4.2}$$

For any $x\in X$, let $\Phi_x: X\to Y$ be such that

$$\Phi_x(z):=\Phi(x,z)\quad\text{for all } z\in X.$$

Then Φ_x is a continuous linear operator from X to Y. Let Φ_x^* denote the conjugate operator of Φ_x, that is,

$$\langle \Phi_x^*(y^*), z\rangle = \langle y^*, \Phi_x(z)\rangle \quad \forall y^* \in Y^* \text{ and } z \in X.$$

It is clear that

$$f'(x) = 2\Phi_x + T \quad \text{and} \quad f''(x) = 2\Phi \quad \forall x \in X, \tag{4.3}$$

where f is as in (4.1). For $x \in P$, let $I(x) := \{j \in [1,k]\colon\ \langle a_j^*, x\rangle - c_j = 0\}$. Then

$$N(P,x) = \left\{ \sum_{j\in I(x)} \lambda_j a_j^*\colon\ \lambda_j \geqslant 0\ \big(j \in I(x)\big)\right\}, \tag{4.4}$$

where $N(P,x)$ denotes the normal cone of P at x (cf. [2] and [19]). In this section, we will consider the optimality conditions for (4.2). The following proposition provides optimality conditions for (4.2) and is essentially known.

Proposition 4.1. *Let $\bar{x} \in P$ and consider the following statements*:

(i) *$\bar{x}$ is a global weak Pareto solution of* (4.2).
(ii) *$\bar{x}$ is a local weak Pareto solution of* (4.2).
(iii) *There exist $c^* \in C^+ \setminus \{0\}$ and $t_j \geqslant 0$ $(j \in I(\bar{x}))$ such that*

$$2\Phi_{\bar{x}}^*(c^*) + T^*(c^*) + \sum_{j\in I(\bar{x})} t_j a_j^* = 0. \tag{4.5}$$

Then (ii) $\Rightarrow$ (iii). *If, in addition, Φ is positively semi-definite, then* (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

Proof. (i) $\Rightarrow$ (ii) is trivial. When Φ is positively semi-definite, $c^* \circ f$ is convex for $c^* \in C^+$; noting that (4.5) means $0 \in (c^* \circ f)'(\bar{x}) + N(P,\bar{x})$, it follows that $\bar{x}$ is a global minimizer of $c^* \circ f$ on P. This shows that (iii) $\Rightarrow$ (i) holds. Take $e \in \operatorname{int}(C)$ such that $e + B_Y \subset C$ (because C is a convex cone with nonempty interior). Then, $C^+ \subset \{y^* \in Y^*\colon\ \|y^*\| \leqslant \langle y^*, e\rangle\}$ and so C is dually compact (cf. [22, Definition 3.1]). Since $f(x) = \Phi(x^2) + T(x) + b$ is smooth and (4.3) holds, similar to the proof of [22, Theorem 4.3], one can prove that (ii) $\Rightarrow$ (iii) holds. □

In contrast to Proposition 4.1, we have the following characterization for ideal solutions of (4.2).

Proposition 4.2. *Let $\bar{x}$ be a point in P and suppose that Φ is positively semi-definite. Then $\bar{x}$ is a global ideal solution of* (4.2) *if and only if for any $c^* \in C^+$ there exist $t_j \geqslant 0$ $(j \in I(\bar{x}))$ such that* (4.5) *holds.*

Proof. First suppose that $\bar{x}$ is an ideal solution of (4.2). Then $f(\bar{x}) \leqslant_C f(x)$ for all $x \in P$ (where f is defined by (4.1)). Let c^* be an arbitrary element in C^+ and

$$\phi_{c^*}(x) = \langle c^*, f(x)\rangle \quad \forall x \in X.$$

Then, $\bar{x}$ is a minimizer of ϕ_{c^*} over P. This implies that $0 \in \phi'_{c^*}(\bar{x}) + N(P,\bar{x})$. Noting that $\phi'_{c^*}(\bar{x}) = (f'(\bar{x}))^*(c^*)$, it follows from (4.3) and (4.4) that there exist $t_j \geqslant 0$ $(j \in I(\bar{x}))$ such that (4.5) holds.

Conversely, take any $c^* \in C^+$. Then, there exist $t_j \geqslant 0$ $(j \in I(\bar{x}))$ such that (4.5) holds. Hence $0 \in \phi_{c^*}(\bar{x}) + N(P,\bar{x})$ (by (4.3) and (4.4)). Noting that ϕ_{c^*} is convex, this means that $\bar{x}$ is a minimizer of ϕ_{c^*} over P. Hence, $\langle c^*, f(\bar{x})\rangle \leqslant \langle c^*, f(x)\rangle$ for any $x \in P$ and $c^* \in C^+$. It follows that $f(\bar{x}) \leqslant_C f(x)$ for any $x \in P$ and so $\bar{x}$ is an ideal solution of (4.2). The proof is completed. □

In contrast to Theorems 3.3 and 3.4, the following theorem shows that every weak Pareto solution of (4.2) is of the global sharp property of fractional order $\frac{1}{2}$.

Theorem 4.3. *Suppose that Φ is S-positively definite with respect to the ordering cone C. Then the following statements hold.*

(i) *There exists a constant $\eta > 0$ such that for every local weak Pareto solution $\bar{x}$ of* (4.2),

$$\eta\|x - \bar{x}\| \leqslant \big(d\big(\Phi(x^2) + T(x) - \Phi(\bar{x}^2) - T(\bar{x}), -C\big)\big)^{\frac{1}{2}} \quad \forall x \in P. \tag{4.6}$$

(ii) *For every local weak Pareto solution $\bar{x}$ of* (4.2) *there exists $\eta(\bar{x}) > 0$ such that*

$$\eta(\bar{x})\|x - \bar{x}\| \leqslant \big(d\big(\Phi(x^2) - T(x) - \Phi(\bar{x}^2) - T(\bar{x}), -C\big) + d(x,P)\big)^{\frac{1}{2}} \quad \forall x \in X. \tag{4.7}$$

(iii) *Every local weak Pareto solution of* (4.2) *is a global Pareto solution of* (4.2).

Proof. Since Φ is S-positively definite with respect to C, there exists $\eta > 0$ such that

$$\Phi(x^2) + \eta\|x\|^2 B_Y \subset C \quad \text{for any } x \in X. \tag{4.8}$$

Let $\bar{x}$ be a local weak Pareto solution of (4.2). Then, by Proposition 4.1, there exists $c^* \in C^+$ with $\|c^*\| = 1$ such that

$$(c^* \circ f)(\bar{x}) = \min\{(c^* \circ f)(x):\ x \in P\} \tag{4.9}$$

where f is defined by (4.1). On the other hand, (4.8) implies that

$$0 \leqslant \inf\{\langle c^*, y\rangle:\ y \in \Phi(x^2) + \eta\|x\|^2 B_Y\} = \langle c^*, \Phi(x^2)\rangle - \eta\|x\|^2 \quad \forall x \in X.$$

Hence

$$\eta\|x\|^2 \leqslant (c^* \circ \Phi)(x^2) \leqslant \tau\|x\|^2 \quad \forall x \in X, \tag{4.10}$$

where $\tau := \max\{\|\Phi(u, v)\|:\ u, v \in B_X\}$ is the norm of the bilinear, symmetric and continuous mapping Φ. Therefore, X is a Hilbert space equipped with the inner product

$$\langle u, v\rangle_e := (c^* \circ \Phi)(u, v) \quad \forall u, v \in X.$$

It follows from the Lax–Milgram theorem that there exists $\theta \in X$ such that

$$-\frac{1}{2}(c^* \circ T)(x) = \langle x, \theta\rangle_e = (c^* \circ \Phi)(\theta, x) \quad \forall x \in X.$$

Hence, for all $x \in X$,

$$(c^* \circ f)(x) = \langle x, x\rangle_e - 2\langle x, \theta\rangle_e + \langle c^*, b\rangle = \|x - \theta\|_e^2 - \|\theta\|_e^2 + \langle c^*, b\rangle,$$

where $\|u\|_e := \langle u, u\rangle_e$ for $u \in X$. Then, for any $x \in X$,

$$(c^* \circ f)(x) - (c^* \circ f)(\bar{x}) = \|x - \theta\|_e^2 - \|\bar{x} - \theta\|_e^2 \tag{4.11}$$
$$= \|x - \bar{x}\|_e^2 + 2\langle x - \bar{x}, \bar{x} - \theta\rangle_e. \tag{4.12}$$

Moreover, by (4.9) and (4.11), one has

$$\|\theta - \bar{x}\|_e = \min\{\|\theta - x\|_e:\ x \in P\}.$$

Applying the projection theorem in the Hilbert space $(X, \|\cdot\|_e)$, it follows that

$$\langle \theta - \bar{x}, x - \bar{x}\rangle_e \leqslant 0 \quad \text{for all } x \in P. \tag{4.13}$$

This and (4.12) imply that

$$(c^* \circ f)(x) - (c^* \circ f)(\bar{x}) \geqslant \|x - \bar{x}\|_e^2 \quad \text{for all } x \in P.$$

By (4.10), one has

$$\eta\|x - \bar{x}\|^2 \leqslant (c^* \circ f)(x) - (c^* \circ f)(\bar{x}) \quad \text{for all } x \in P. \tag{4.14}$$

On the other hand, since $c^* \in C^+$ and $\|c^*\| = 1$, one has

$$\begin{aligned}(c^* \circ f)(x) - (c^* \circ f)(\bar{x}) &= \langle c^*, \Phi(x^2) + T(x) - \Phi(\bar{x}^2) - T(\bar{x})\rangle \\ &\leqslant \langle c^*, \Phi(x^2) + T(x) - \Phi(\bar{x}^2) - T(\bar{x}) + c\rangle \\ &\leqslant \|\Phi(x^2) + T(x) - \Phi(\bar{x}^2) - T(\bar{x}) + c\|\end{aligned}$$

for any $x \in X$ and any $c \in C$. Hence

$$(c^* \circ f)(x) - (c^* \circ f)(\bar{x}) \leqslant d(\Phi(x^2) + T(x) - \Phi(\bar{x}^2) - T(\bar{x}), -C) \quad \forall x \in X. \tag{4.15}$$

This and (4.14) imply that (4.6) holds. This completes the proof of (i).

Similar to the proof of the corresponding part of Theorem 3.3, we can see that (iii) is immediate from (i).

To prove (ii), we consider two cases: C1) $\bar{x} = \theta$ and C2) $\bar{x} \neq \theta$. First suppose that C1) holds. Then, by (4.11) and (4.15), one has

$$\|x - \bar{x}\|_e^2 \leqslant d(\Phi(x^2) + T(x) - \Phi(\bar{x}^2) - T(\bar{x}), -C) \quad \forall x \in X.$$

This and (4.10) imply that (4.7) holds with $\eta(\bar{x}) = \eta^{\frac{1}{2}}$. Next suppose that C2) holds. Let

$$v := \frac{\theta - \bar{x}}{\|\bar{x} - \theta\|_e} \quad \text{and} \quad H := \{z \in X\colon \langle v, z\rangle_e \leqslant \langle v, \bar{x}\rangle_e\}.$$

Then, by (4.13), one has $P \subset H$. For any $x \in X$, letting $d_{\|\cdot\|_e}(x, P) := \inf\{\|x - z\|_e\colon z \in P\}$ and noting that

$$\langle v, x - \bar{x}\rangle_e \leqslant \langle v, x - z\rangle_e \leqslant \|v\|_e \|x - z\|_e = \|x - z\|_e \quad \forall z \in P,$$

one has $\langle v, x - \bar{x}\rangle_e \leqslant d_{\|\cdot\|_e}(x, P)$. Letting $\beta := 2\|\theta - \bar{x}\|_e$, it follows from (4.12) that

$$\beta d_{\|\cdot\|_e}(x, P) + (c^* \circ f)(x) - (c^* \circ f)(\bar{x}) \geqslant \beta\langle v, x - \bar{x}\rangle_e + \|x - \bar{x}\|_e^2 + 2\langle x - \bar{x}, \bar{x} - \theta\rangle_e = \|x - \bar{x}\|_e^2.$$

This and (4.15) imply that

$$\|x - \bar{x}\|_e^2 \leqslant d(\Phi(x^2) + T(x) - \Phi(\bar{x}^2) - T(\bar{x}), -C) + \beta d_{\|\cdot\|_e}(x, P) \quad \forall x \in X.$$

Noting that $\eta\|x - \bar{x}\|^2 \leqslant \|x - \bar{x}\|_e^2$ and $d_{\|\cdot\|_e}(x, P) \leqslant \tau^{\frac{1}{2}} d(x, P)$ (by (4.10)), this implies that (4.7) holds with $\eta(\bar{x}) = (\frac{\eta}{1+\beta\tau^{\frac{1}{2}}})^{\frac{1}{2}}$. The proof is completed. □

From Proposition 4.2 and Theorem 4.3, it is easy to verify the following result on ideal solutions.

Theorem 4.4. *Let $\bar{x}$ be a point in the feasible set P and suppose that Φ is S-positively definite. Then, the following statements are equivalent.*

(i) *$\bar{x}$ is an ideal solution of* (4.2) *and there exists a constant $\eta > 0$ such that*

$$\eta\|x - \bar{x}\| \leqslant \big(d(\Phi(x^2) + T(x) - \Phi(\bar{x}^2) - T(\bar{x}), -C)\big)^{\frac{1}{2}} \quad \forall x \in P.$$

(ii) *$\bar{x}$ is a local ideal solution of* (4.2).

(iii) *For any $c^* \in C^+$ there exist $t_j \geqslant 0$ $(j \in I(\bar{x}))$ such that* (4.5) *holds.*

In what follows, let S_w and S denote the set of all weak Pareto solutions of (4.2) and the set of all Pareto solutions of (4.2), respectively. Deng and Yang [4] considered weak sharp minima for a linear vector optimization problem in Euclidean spaces and proved the following result: *Suppose that $X = \mathbb{R}^m$, $Y = \mathbb{R}^n$, $C = \mathbb{R}^n_+$ and $\Phi = 0$. Further suppose that $S_w \neq \emptyset$. Then there exists $\eta > 0$ such that*

$$\eta d(x, S_w) \leqslant \inf_{\bar{x}\in S_w} \|T(x) - T(\bar{x})\| \quad \forall x \in P.$$

It is possible that $S_w = \emptyset$ in the case when $X = \mathbb{R}^m$, $Y = \mathbb{R}^n$, $C = \mathbb{R}^n_+$ and $\Phi = 0$. Under the S-positive definiteness assumption on Φ, we will prove that $S_w \neq \emptyset$ and Deng and Yang's result holds for a general quadratic vector optimization problem in Banach spaces.

Theorem 4.5. *Suppose that Φ is S-positively definite with respect to C. Then*

$$S_w = S \neq \emptyset \tag{4.16}$$

and there exists $\eta > 0$ such that

$$\eta d(x, S_w) \leqslant \inf_{\bar{x}\in S_w} d(\Phi(x^2) + T(x) - \Phi(\bar{x}^2) - T(\bar{x}), -C)^{\frac{1}{2}} \quad \forall x \in P. \tag{4.17}$$

Proof. To prove (4.16), by Theorem 4.3(iii), it suffices to show that $S_w \neq \emptyset$. Take an element c^* in C^+ with $\|c^*\| = 1$. Since Φ is S-positively definite, there exist $\eta, \tau \in (0, +\infty)$ such that (4.10) holds. Let $\langle\cdot,\cdot\rangle_e$ and $\|\cdot\|_e$ be as in the proof of Theorem 4.4. Then $(X, \langle\cdot,\cdot\rangle_e)$ is a Hilbert space and (4.11) implies that

$$(c^* \circ f)(x) = \|x - \theta\|_e^2 + r \quad \forall x \in X, \tag{4.18}$$

where θ is a fixed point in X and $r = (c^* \circ f)(\bar{x}) - \|\bar{x} - \theta\|_e^2$ is a constant independent of x. Since the feasible set P of (4.2) is a closed and convex subset of X, the projection theorem implies that there exists $\bar{x} \in P$ such that

$$\|\theta - \bar{x}\|_e = d_{\|\cdot\|_e}(\theta, P) := \inf\{\|\theta - x\|_e\colon x \in P\}.$$

It follows from (4.18) that

$$\big(c^* \circ f\big)(\bar{x}) \leqslant \big(c^* \circ f\big)(x) \quad \forall x \in P.$$

This implies that $\bar{x} \in S_W$ and so (4.16) holds. By Theorem 4.3(i), there exists $\eta > 0$ such that (4.6) holds for any $\bar{x} \in S_W$. Hence

$$\inf_{\bar{x}\in S_W} \eta\|x-\bar{x}\| \leqslant \inf_{\bar{x}\in S_W} d\big(\Phi(x,x)+T(x)-\Phi(\bar{x},\bar{x})-T(\bar{x}),-C\big)^{\frac{1}{2}} \quad \forall x \in P,$$

which means that (4.17) holds. The proof is completed. □

The following example shows that the S-positive definiteness assumption on Φ cannot be dropped in Theorem 4.5.

Example 4.1. Let $X=\mathbb{R}^3$, $Y=\mathbb{R}^4$, $C=\mathbb{R}^4_+$ and $P=\mathbb{R}^2\times\{0\}$; let $\Phi: X\times X\to\mathbb{R}^4$ and $T: X\to\mathbb{R}^4$ be defined by

$$\Phi\big((u_1,u_2,u_3),(v_1,v_2,v_3)\big)=(0,u_1v_1,0,0) \quad \forall (u_1,u_2,u_3),(v_1,v_2,v_3)\in X,$$

and

$$T(u_1,u_2,u_3)=(u_1,0,u_2,u_3) \quad \forall (u_1,u_2,u_3)\in X,$$

respectively; let $b=(0,0,0,0)$. Then Φ is positively semi-definite and the objective function is

$$f(u,v,w)=\big(u,u^2,v,w\big) \quad \forall (u,v,w)\in X.$$

Hence $f(P)=\{(u,u^2,v,0)\colon u,v\in\mathbb{R}\}$. It is easy to verify that

$$\mathrm{WE}\big(f(P),C\big)=\big\{\big(u,u^2,v,0\big)\colon (u,v)\in(-\mathbb{R}_+)\times\mathbb{R}\big\} \quad \text{and} \quad \mathrm{E}\big(F(P),C\big)=\emptyset.$$

It follows that

$$S_W=f^{-1}\big(\mathrm{WE}\big(f(P),C\big)\big)\cap P=(-\mathbb{R}_+)\times\mathbb{R}\times\{0\} \quad \text{and} \quad S=f^{-1}\big(\mathrm{E}\big(f(P),C\big)\big)\cap P=\emptyset.$$

Let $u\in(0,+\infty)$ and $x_u:=(u,0,0)$. Then, $x_u\in P$, $d(x_u,S_W)=u$ and $-x_u\in S_W$. Noting that $\Phi(x_u^2)+T(x_u)-\Phi((-x_u)^2)-T(-x_u)=(2u,0,0,0)$, it follows that

$$d\big(\Phi\big(x_u^2\big)+T(x_u)-\Phi\big((-x_u)^2\big)-T(-x_u),-C\big)\leqslant 2u$$

and so

$$\inf_{\bar{x}\in S_W} d\big(\Phi\big(x_u^2\big)+T(x_u)-\Phi\big(\bar{x}^2\big)-T(\bar{x}),-C\big)\leqslant 2u.$$

Hence

$$\lim_{u\to+\infty}\frac{\inf_{\bar{x}\in S_W} d(\Phi(x_u,x_u)+T(x_u)-\Phi(\bar{x},\bar{x})-T(\bar{x}),-C)}{d(x_u,S_W)^2}=0.$$

This shows that (4.17) does not hold.

Let S_i denote the set of all ideal solutions of (4.2). In contrast to Theorem 4.5, S_i may be empty under the S-positive definiteness assumption on Φ. Indeed, let $X=\mathbb{R}$, $Y=\mathbb{R}^2$, $C=\mathbb{R}^2_+$ and $P=\mathbb{R}$; let

$$\Phi(u,v):=(uv,uv) \quad \text{and} \quad T(u):=(0,-2u) \quad \forall (u,v)\in\mathbb{R}^2,$$

and let $b=(0,1)$. Then Φ is S-positively definite and the objective function $f(x)=(x^2,(x-1)^2)$ for all $x\in\mathbb{R}$. Letting $(s,t)\in\mathbb{R}^2$, it is clear that

$$(s,t)\leqslant_{\mathbb{R}^2_+} f(x) \quad \forall x\in\mathbb{R} \quad \Leftrightarrow \quad (s,t)\in-\mathbb{R}^2.$$

Noting that $f(x)\notin-\mathbb{R}^2$ for all $x\in\mathbb{R}$, it follows that $S_i=\emptyset$.

Let Q_i be an $n\times n$ positively semi-definite matrix, $u_i\in\mathbb{R}^n$ and $r_i\in\mathbb{R}$ $(i=1,\ldots,m)$. Let P be a polyhedron in $\mathbb{R}^n$ and consider the following quadratic convex multi-objective optimization problem:

$$\min_{x\in P}\big(x^TQ_1x+u_1^Tx+r_1,\ldots,x^TQ_mx+u_m^Tx+r_m\big). \tag{4.19}$$

We say that $\bar{x}\in P$ is a weak Pareto solution of (4.19) if there exists no $x\in P$ such that

$$x^TQ_1x+u_i^Tx+r_i<\bar{x}^TQ_i\bar{x}+u_i^T\bar{x}+r_i \quad \text{for each } i\in[0,m].$$

Let $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$ and $C = \mathbb{R}^m_+$. Let $\Phi : X \times X \to Y$ and $T : X \to Y$ be respectively defined by $\Phi(u,v) = (u^T Q_1 v, \ldots, u^T Q_m v)$ and $T(x) = (u_1^T x, \ldots, u_m^T x)$ for all $(u,v) \in X \times X$ and $x \in X$, let $b := (r_1, \ldots, r_m)$. Clearly, Φ is a bilinear, symmetric, continuous and positively semi-definite mapping with respect to the ordering cone $\mathbb{R}^n_+$, and it is easy from Proposition 3.2 to verify that Φ is S-positively definite with respect to $\mathbb{R}^n_+$ if and only if each Q_i is a positively definite matrix. Noting that $\Phi^*_{\bar{x}}(c^*) = (\bar{x}^T Q_1 c^*, \ldots, \bar{x}^T Q_m c^*)$, the following proposition is an immediate consequence of Theorems 4.4 and 4.5.

Proposition 4.6. *Let $\bar{x}$ be a point in P. Then the following statements hold.*

(i) *$\bar{x}$ is a local weak Pareto solution of* (4.19) *if and only if there exist $s_i \geqslant 0$ $(1 \leqslant i \leqslant n)$ and $t_j \geqslant 0$ $(j \in I(\bar{x}))$ such that some s_i is not zero and*

$$\sum_{i=1}^{n} s_i(Q_i\bar{x} + u_i) + \sum_{j \in I(\bar{x})} t_j a_j^* = 0. \tag{4.20}$$

(ii) *$\bar{x}$ is a global ideal solution of* (4.19) *if and only if for any $s_i \geqslant 0$ $(1 \leqslant i \leqslant n)$ there exist $t_j \geqslant 0$ $(j \in I(\bar{x}))$ such that* (4.20) *holds.*

(iii) *If each Q_i is a positively definite matrix, then every weak Pareto solution of* (4.19) *is a Pareto solution of* (4.19) *and there exists a constant $\eta > 0$ such that*

$$\eta\|x - \bar{x}\| \leqslant \max_{1 \leqslant i \leqslant m} \left[x^T Q_i x + u_i^T x - \bar{x}^T Q_i \bar{x} - u_i^T \bar{x}\right]_+^{\frac{1}{2}} \quad \forall \bar{x} \in S_w \text{ and } \forall x \in P.$$

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References

[1] G.Y. Chen, X.X. Huang, X.Q. Yang, Vector Optimization: Set-Valued and Variational Analysis, Lecture Notes in Econom. and Math. Systems, vol. 541, Springer-Verlag, Berlin, 2005.
[2] F.H. Clarke, Optimization and Nonsmooth Analysis, Wiley, New York, 1983.
[3] L. Cromme, Strong uniqueness, a far-reaching criterion for the convergence analysis of iterative procedures, Numer. Math. 29 (1978) 179–193.
[4] S. Deng, X.Q. Yang, Weak sharp minima in multicriteria programming, SIAM J. Optim. 15 (2004) 456–460.
[5] A. Gopfert, H. Riahi, C. Tammer, C. Zalinescu, Variational Methods in Partially Ordered Spaces, Springer-Verlag, New York, 2003.
[6] M.C. Ferris, Weak sharp minima and penalty functions in mathematical programming, PhD thesis, University of Cambridge, Cambridge, 1988.
[7] A. Gotz, J. Jahn, The Lagrange multiplier rule in set-valued optimization, SIAM J. Optim. 10 (1999) 331–344.
[8] C. Gutierrez, E. Jimenez, V. Novo, A unified approach and optimality conditions for approximate solutions of vector optimization problems, SIAM J. Optim. 17 (2006) 688–710.
[9] J. Jahn, Vector Optimization: Theory, Applications and Extensions, Springer-Verlag, 2004.
[10] B. Jimenez, Strict efficiency in vector optimization, J. Math. Anal. Appl. 265 (2002) 264–284.
[11] B. Jimenez, Strict minimality conditions in nondifferentiable multiobjective programming, J. Optim. Theory Appl. 116 (2003) 99–116.
[12] K. Jittorntrum, M.R. Osborne, Strong uniqueness and second order convergence in nonlinear discrete approximation, Numer. Math. 34 (1980) 439–455.
[13] D.T. Luc, Theory of Vector Optimization, Springer-Verlag, Berlin, Heidelberg, 1989.
[14] D.T. Luc, Taylor's formula for $C^{k,1}$ functions, SIAM J. Optim. 5 (3) (1995) 659–669.
[15] B.S. Modukhovich, J.S. Treiman, Q.J. Zhu, An extended extremal principle with applications to multiobjective optimization, SIAM J. Optim. 14 (2003) 359–379.
[16] M.R. Osborne, R.S. Womersley, Strong uniqueness in sequential linear programming, J. Austral. Math. Soc. Ser. B 31 (1990) 379–384.
[17] B.T. Polyak, Introduction to Optimization, Optimization Software, New York, 1987.
[18] R.T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, NJ, 1970.
[19] R.T. Rockafellar, R.J.-B. Wets, Variational Analysis, Springer, New York, 1998.
[20] J.J. Ye, Q.J. Zhu, Multiobjective optimization problem with variational inequality constraints, Math. Program. 96 (2003) 139–160.
[21] X.Y. Zheng, X.M. Yang, K.L. Teo, Sharp minima for multiobjective optimization in Banach spaces, Set-Valued Anal. 14 (2006) 327–345.
[22] X.Y. Zheng, K.F. Ng, The Fermat rule for multifunctions on Banach spaces, Math. Program. 104 (2005) 69–90.